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2009 J. Phys. A: Math. Theor. 42 375201

(http://iopscience.iop.org/1751-8121/42/37/375201)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **42** (2009) 375201 (19pp) doi:10.1088/1751-8113/42/37/375201

On the constructive determination of the periodic potentials from the Bloch eigenvalues

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Received 11 February 2009
Published 25 August 2009
Online at stacks.iop.org/JPhysA/42/375201

Abstract

In this paper, we consider the three-dimensional Schrödinger operator with a periodic, relative to a lattice Ω of $\mathbb{R}^3$, potential q. We construct a set D of trigonometric polynomials such that

(a) D is dense in $W_2^s(\mathbb{R}^3/\Omega)$, where $s > 3$, in the $\mathbb{C}^\infty$-topology,

(b) any element q of the set D can be determined constructively and uniquely, modulo inversion and translation $q(x) \to q(-x)$, $q(x) \to q(x+\tau)$, where $\tau \in \mathbb{R}^3$, from the given Bloch eigenvalues of the Schrödinger operator with the potential q.

PACS numbers: 02.30.Jr, 02.30.Tb, 02.30.Zz

1. Introduction

We investigate the inverse problem for the three-dimensional Schrödinger operator $L(q)$ generated in $L_2(\mathbb{R}^3)$ by the differential expression

$$l(u) = -\Delta u + q(x)u, \quad \text{where} \quad x \in \mathbb{R}^3,$$

with a real periodic, relative to a lattice Ω of $\mathbb{R}^3$, potential $q(x)$. Let $\omega_1, \omega_2, \omega_3$ be a basis of the lattice Ω and

$$F = \{c_1\omega_1 + c_2\omega_2 + c_3\omega_3 : c_k \in [0, 1), k = 1, 2, 3\}$$

be a fundamental domain $\mathbb{R}^3/\Omega$ of Ω. The spectrum of $L(q)$ is the union of the spectra of operators $L_t(q)$ for $t \in F^*$ generated in $L_2(F)$ by the expression $l(u)$ and the conditions

$$u(x + \omega) = \mathrm{e}^{\mathrm{i}\langle t,\omega\rangle} u(x), \qquad \forall \omega \in \Omega,$$

where F^* is the fundamental domain of the lattice Γ, Γ is the lattice dual to Ω and $\langle .,.\rangle$ is the inner product in $\mathbb{R}^3$. The eigenvalues $\Lambda_1(t) \leqslant \Lambda_2(t) \leqslant \cdots$ of $L_t(q)$ are called the Bloch eigenvalues of $L(q)$. These eigenvalues define the functions $\Lambda_1(t), \Lambda_2(t), \ldots$ of t that are called the band functions of $L(q)$. The aim of this paper is the constructive determination of the potential q of the three-dimensional Schrödinger operator $L(q)$ from the band functions.

Eskin, Ralston and Trubowitz [2, 3] proved the following result about the inverse problem of the two-dimensional Schrödinger operator $L(q)$.

For $\Omega \subset \mathbb{R}^2$ satisfying the condition: if $|\omega'| = |\omega|$ for $\omega, \omega' \in \Omega$, then $\omega' = \pm\omega$; there is a set $\{M_\alpha\}$ of manifolds of potentials such that

(a) $\{M_\alpha\}$ is dense in the set of smooth periodic potentials in the $\mathbb{C}^\infty$-topology,

(b) for each α there is a dense open set $Q_\alpha \subset M_\alpha$ such that for $q \in Q_\alpha$ the set of real analytic $\widetilde{q}$ satisfying

$$\mathrm{Spec}(L_0(q)) = \mathrm{Spec}(L_0(\widetilde{q}))$$

and the set of $\widetilde{q} \in \mathbb{C}^6(F)$ satisfying

$$\mathrm{Spec}(L_t(q)) = \mathrm{Spec}(L_t(\widetilde{q}))$$

for all $t \in \mathbb{R}^2$ are finite modulo translations, where $\mathrm{Spec}(L_t(q))$ is the spectrum of $L_t(q)$.

In this paper, we give an algorithm for the unique (modulo inversion and translation) determination of the potential q of the three-dimensional Schrödinger operator $L(q)$ from the spectral invariants which were determined constructively in [4] from the given band functions. As a result, we determine constructively the potential from the given band functions.

To describe the brief scheme of this paper, we begin by recalling the invariants obtained in [4] which will be used here. Let a be a visible element of Γ, that is, a is an element of Γ of the minimal norm belonging to the line $a\mathbb{R}$, and

$$q^a(x) = \sum_{n\in\mathbb{Z}} (q(x), \mathrm{e}^{\mathrm{i}n\langle a,x\rangle})\mathrm{e}^{\mathrm{i}n\langle a,x\rangle}$$

be the directional (one-dimensional) potential, where $(.\,,\,.)$ is the inner product in $L_2(F)$. Let Ω_a be the sublattice $\{d \in \Omega : \langle d, a\rangle = 0\}$ of Ω in the hyperplane $H_a = \{x \in \mathbb{R}^3 : \langle x, a\rangle = 0\}$ and Γ_a be the lattice dual to Ω_a. Let β be a visible element Γ_a and $P(a, \beta)$ be the plane containing a, β, and the origin. Define a function $q_{a,\beta}(x)$ by

$$q_{a,\beta}(x) = \sum_{c\in(P(a,\beta)\cap\Gamma)\backslash a\mathbb{R}} \frac{c}{\langle\beta, c\rangle} z(c)\mathrm{e}^{\mathrm{i}\langle c,x\rangle}, \tag{1}$$

where $z(c) \equiv (q(x), \mathrm{e}^{\mathrm{i}\langle c,x\rangle})$ for $c \in \Gamma$ is the Fourier coefficient of q. In [4], we constructively determined the following spectral invariants:

$$\int_F |q^a(x)|^2 \mathrm{d}x, \tag{2}$$

$$\int_F |q_{a,\beta}(x)|^2 q^a(x)\mathrm{d}x \tag{3}$$

from the asymptotic formulae for the band functions of $L(q)$ obtained in [5, 6]. Moreover, in [4] we constructively determined the invariant

$$\int_F |q_{a,\beta}(x)|^2 (z^2(a)\mathrm{e}^{\mathrm{i}2\langle a,x\rangle} + z^2(-a)\mathrm{e}^{-\mathrm{i}2\langle a,x\rangle})\mathrm{d}x \tag{4}$$

when the directional potential $q^a(x)$ has the form

$$q^a(x) = z(a)\mathrm{e}^{\mathrm{i}\langle a,x\rangle} + z(-a)\mathrm{e}^{-\mathrm{i}\langle a,x\rangle}. \tag{5}$$

In this paper, fixing the inversion and translation

$$q(x) \to q(-x), \qquad q(x) \to q(x+\tau), \qquad \tau \in \mathbb{R}^3, \tag{6}$$

we give an algorithm for the unique determination of the potential q of the three-dimensional Schrödinger operator $L(q)$ from the invariants (2)–(4). Note that the potential q can uniquely be determined only by fixing the inversion and translation (6), since $L(q(x))$, $L(q(-x))$, $L(q(x+\tau))$ have the same band functions and hence the same invariants (2)–(4).

First, we consider the invariants (2)–(4) for the trigonometric polynomials of the form

$$q(x) = \sum_{a\in Q(N,M,S)} z(a)\mathrm{e}^{\mathrm{i}\langle a,x\rangle}, \tag{7}$$

where N, M, S are integers,

$$Q(N, M, S) = \{n\gamma_1 + m\gamma_2 + s\gamma_3 : |n| \leqslant N, |m| \leqslant M, |s| \leqslant S\}\backslash\{0\},$$

and $\{\gamma_1, \gamma_2, \gamma_3\}$ is a basis of Γ satisfying $\langle\gamma_i, \omega_j\rangle = 2\pi\delta_{i,j}$. If $a = n\gamma_1 + m\gamma_2 + s\gamma_3$, then we write (n, m, s) and$z(n, m, s)$ instead of a and $z(a)$, respectively. For brevity of the notations, instead of $Q(N, M, S)$ we write Q if it is not ambiguous.

To describe the invariants (2)–(4) for (7), let us introduce some notations. If $b \in (\Gamma \cap P(a, \beta))\backslash a\mathbb{R}$, then the plane $P(a, \beta)$ coincides with the plane $P(a, b)$. Moreover, every vector $b \in (P(a, \beta) \cap \Gamma)\backslash a\mathbb{R}$ has an orthogonal decomposition (see (20) in [5])

$$b = s\beta + \mu a, \tag{8}$$

where s is a nonzero integer, β is a visible element of Γ_a and μ is a real number. Therefore, for every plane $P(a, b)$, where $b \in \Gamma$, there exists a plane $P(a, \beta)$, where β is as defined by (8), which coincides with $P(a, b)$.

Notation 1. *For every pair* $\{a, b\}$*, where a is a visible element of* Γ *and* $b \in \Gamma$*, we denote by* $I_1(a, b)$ *and* $I_2(a, b)$ *the invariants (3) and (4), respectively, where* β *is a visible element of* Γ_a *defined by (8).*

Definition 1. *A visible vector* $a \in \Gamma$ *is said to be long visible (with respect to* Q*) if* $sa \in Q$ *if and only if* $s = \mp 1$.

If a is long visible, then the directional potential q^a of (7) has the form (5). Therefore the invariant (2) is

$$\|q^a\|^2 \equiv 2|z(a)|^2, \tag{9}$$

and hence the invariant (2) gives the absolute value of the Fourier coefficient $z(a)$. Moreover, we prove that there exist a lot of pairs $\{a, b\}$ such that the invariants (9), $I_1(a, b)$ and $I_2(a, b)$ give the following simple invariants,

$$S_1(a, b) = Re(z(-a)z(a - b)z(b)), \qquad A_1(a, b) = \cos(-\alpha(a) + \alpha(a - b) + \alpha(b)), \tag{10}$$

$$S_2(a, b) = Re(z^2(-a)z(a + b)z(a - b)), \quad A_2(a, b) = \cos(-2\alpha(a) + \alpha(a + b) + \alpha(a - b)), \tag{11}$$

where $\mathrm{Re}(z)$ is the real part of the complex number z, $z(a) = r(a)\mathrm{e}^{\mathrm{i}\alpha(a)}$, $\alpha(a) \in (-\pi, \pi]$. In other words, for these pairs we have the equations,

$$-\alpha(a) + \alpha(a - b) + \alpha(b) = d(a, b)e(1, a, b)(\mathrm{mod}2\pi), \tag{12}$$

$$-2\alpha(a) + \alpha(a + b) + \alpha(a - b) = d(a, b)e(2, a, b)(\mathrm{mod}2\pi), \tag{13}$$

where $e(i, a, b) =: \arccos A_i(a, b)$ for $i = 1, 2$ are the known numbers belonging to $[0, \pi]$, $d(a, b) = \pm 1$, and the equality $\theta = \varphi(\mathrm{mod} 2\pi)$ means that $\theta - \varphi = 2k\pi$ for some integer k.

In section 2, we consider the invariants (3), (4) for the polynomials (7) and find a lot of pairs $\{a, b\}$ such that there exist the simple invariants $A_1(a, b)$, $A_2(a, b)$ corresponding to these pairs.

In section 3, we give an algorithm for finding the Fourier coefficients $z(n, m, s)$ when $(n, m, s) \in B(N, M, S)$, where

$$B(N, M, S) = \{(n, m, s) \in Q(N, M, S) : nms(|n| - N)(|m| - M)(|s| - S) = 0\}.$$

First, we find $z(a)$ when a belongs to the boundary $\partial \widetilde{Q}$ of the parallelepiped

$$\widetilde{Q} =: \{x = (x_1, x_2, x_3) : |x_1| \leqslant N, |x_2| \leqslant M, |x_2| \leqslant S\},$$

that is, we find the Fourier coefficients $z(n, m, s)$ if either $n = N, -N$, or $m = M, -M$, or $s = S, -S$. For this, we use the following two observations.

(1) All boundary points of $\widetilde{Q}$ except the points of the set

$$A(N, M, S) = \{(\pm N, 0, 0), (0, \pm M, 0), (0, 0, \pm S)\} \cup \{(n, m, s) : |n| = |m| = |s| = N\}$$

are long visible, if N, M, S are distinct prime numbers, satisfying $N < M < S$. Hence, the absolute value $r(a)$ of $z(a)$ is known by (9).

(2) If a is a boundary point of $\widetilde{Q}$, then there are a lot of vectors b such that there exists a simple invariant $A_2(a, b)$ corresponding to the pair $\{a, b\}$.

Thus, we can write a lot of equations of type (13) with respect to the argument of the Fourier coefficients. If $d(a, b)$ and the values of two summands in the left-hand side of (13) are known, then one can find the value of the third summand. To use these equations, we need to know the values of the arguments of some Fourier coefficients. Three of them can be determined by fixing the translation $q(x) \to q(x + \tau)$, that is, by taking one of the functions $q(x + \tau)$. Namely, in section 3, we prove that the conditions

$$\alpha_\tau(N - 1, M, S) = \alpha_\tau(N, M - 1, S) = \alpha_\tau(N, M, S - 1) = 0, \tag{14}$$

$$\alpha_\tau(N, M, S) \in \left[0, \frac{2\pi}{N + M + S - 1}\right), \tag{15}$$

where

$$\alpha_\tau(a) = \arg(q(x + \tau), \mathrm{e}^{\mathrm{i}\langle a, x\rangle}),$$

determine a unique value of τ.

Thus, in section 3, using (14) and a lot of equations of type (13) we determine $z(a)$ when $a \in \partial \widetilde{Q}$. Then, using this, we find $z(n, m, s)$, when $nms = 0$. In section 4, we construct a dense in $W_2^s(F)$, where $s > 3$, in the $\mathbb{C}^\infty$-topology set D of trigonometric polynomials, such that every $q \in D$ can be found by the algorithm given in this paper.

In forthcoming papers, we will give a scheme for the construction of smooth potentials.

2. On the simple invariants

First, let us consider the invariants (3), (4) for the trigonometric polynomial (7).

Definition 2. *A pair $\{a, b\}$, where a is a long visible element of Q and $b \in Q$, is said to be a canonical pair of type 1 if $\langle b, a - b\rangle \neq 0$ and the following implication holds*

$$\{c, a - c\} \subset (P(a, b) \cap Q)\backslash a\mathbb{R} \Leftrightarrow c \in \{b, a - b\}. \tag{16}$$

A pair $\{a, b\}$, *where* a *is a long visible element of* Q *and* $b \in Q$, *is said to be a canonical pair of type 2 if* $\langle a+b, a-b\rangle \neq 0$ *and the following implication holds:*

$$\{a+c, a-c\} \subset (P(a,b) \cap Q)\backslash a\mathbb{R} \Leftrightarrow c \in \{\pm b\}. \tag{17}$$

Theorem 1. *If the potential* $q(x)$ *has the form* (7) *and* a *is a long visible element of* Q, *then the invariants* $I_1(a,b)$, $I_2(a,b)$, *defined in notation 1, yield the invariants*

$$\mathrm{Re}\left(z(-a)\left(\sum_{c\in G_1} g(a,c)z(a-c)z(c)\right)\right), \tag{18}$$

$$\mathrm{Re}\left(z^2(-a)\left(\sum_{c\in G_2} h(a,c)z(a-c)z(a+c)\right)\right), \tag{19}$$

where

$$g(a,c) = \frac{\langle c, c-a\rangle}{(\langle c,\beta\rangle)^2}, \qquad h(a,c) = \frac{\langle c+a, c-a\rangle}{(\langle c,\beta\rangle)^2},$$

G_1 *and* G_2 *are the set of all* c *such that* $\{c, a-c\} \subset (P(a,b)\cap Q)\backslash a\mathbb{R}$ *and* $\{a+c, a-c\} \subset (P(a,b)\cap Q)\backslash a\mathbb{R}$, *respectively.*

If $\{a,b\}$ *is a canonical pair of type* k, *where* $k = 1, 2$, *then* (18), (19) *give the simple invariants* $S_k(a,b)$, $A_k(a,b)$ *defined in* (10) *and* (11).

Proof. If the potential $q(x)$ has the form (7), then (1) becomes

$$q_{a,\beta}(x) = \sum_{c\in(P(a,\beta)\cap Q)\backslash a\mathbb{R}} \frac{c}{\langle\beta, c\rangle} z(c)\mathrm{e}^{\mathrm{i}\langle c,x\rangle}. \tag{20}$$

Using this and (5) in (3), we obtain

$$I_1(a,b) = \int_F |q_{a,\beta}(x)|^2 q^a(x)\,\mathrm{d}x = \Sigma_1 + \Sigma_2, \tag{21}$$

where $I_1(a,b)$ is as defined in notation 1,

$$\Sigma_1 = \sum_{c\in(P(a,b)\cap Q)\backslash a\mathbb{R}} \frac{\langle c, c+a\rangle}{\langle c,\beta\rangle\,\langle c+a,\beta\rangle} z(c)z(-a-c)z(a)), \tag{22}$$

$$\Sigma_2 = \sum_{c\in(P(a,b)\cap Q)\backslash a\mathbb{R}} \frac{\langle c, c-a\rangle}{\langle c,\beta\rangle\,\langle c-a,\beta\rangle} z(c)z(a-c)z(-a)).$$

Since $Q(N,M,S)$ is symmetric with respect to the origin, the substitution $c' = -c$ in (22) does not change Σ_1. Using this substitution in (22) and then taking into account that $z(-b) = \overline{z(b)}$, $\langle a,\beta\rangle = 0$, we obtain

$$\Sigma_1 = \overline{\Sigma_2}, \qquad \Sigma_1 + \Sigma_2 = \mathrm{Re}(2\Sigma_2).$$

This with (21) shows that the invariant $I_1(a,b)$ gives the invariant (18).

Replacing a by $2a$, in the same way, we obtain the invariant

$$\mathrm{Re}\left(z^2(-a)\left(\sum_{c\in G} \frac{\langle c, c-2a\rangle}{(\langle c,\beta\rangle)^2} z(2a-c)z(c)\right)\right) \tag{23}$$

from the invariant $I_2(a,b)$, where G is the set of all c such that $\{c, 2a-c\} \subset (P(a,b)\cap Q)\backslash a\mathbb{R}$, $\langle c, c-2a\rangle \neq 0$. Thus, in (23) replacing c by $a+c$ and using the obvious equality $\langle a,\beta\rangle = 0$, we get (19).

Now suppose that $\{a, b\}$ is a canonical pair of type 1. Then it follows from the definition of G_1 and from the definition of the canonical pair of type 1 that $G_1 = \{b, a-b\}$. Therefore, (18) has the form

$$\mathrm{Re}\left(z(-a)\left(\frac{\langle b, b-a\rangle}{(\langle b,\beta\rangle)^2}z(a-b)z(b)+\frac{\langle a-b,-b\rangle}{(\langle a-b,\beta\rangle)^2}z(b)z(a-b)\right)\right).$$

The invariant $S_1(a, b)$ can be obtained from this invariant, because $\langle b, b-a\rangle = \langle a-b, -b\rangle$ and $\langle a, \beta\rangle = 0$. The invariant (9) and $S_1(a, b)$ imply $A_1(a, b)$. In the same way, we obtain the invariants $S_2(a, b)$ and $A_2(a, b)$ from (19). □

Now we determine a lot of canonical pairs of types 1 and 2.

Condition 1. *Suppose* $\Gamma = \mathbb{Z}^3$ *and* $z(n, m, s) \neq 0$ *for* $(n, m, s) \in B(N, M, S)$, *where* N, M, S *are prime numbers satisfying* $S > 2M$, $M > 2N$, $N \gg 1$.

Proposition 1. *Suppose condition 1 holds.*

(a) *The pair* $\{a, b\}$ *is a canonical pair of type 2 in each of the following cases:*
   - (1) $a = (N, M-1, s)$, $b = (0, \pm 1, p)$, *where* $s+p, s-p \in [-S, S]$, $|p| \leqslant M-1$.
   - (2) $a = (N, m, S-1)$, $b = (0, q, \pm 1)$, *where* $m+q, m-q \in [-M, M]$.
   - (3) $a = (N, m, s)$, $b = (0, \pm 1, p)$, *where* $m \in [-M+1, M-1]$, $s+p, s-p \in [-S, S]$, $s-2p \notin [-S, S]$, $(N, m, s) \notin A(N, M, S)$ *and* $N^2+m^2-1+s^2-p^2 \neq 0$.

(b) *The pair* $\{a, b\}$ *is a canonical pair of type 1 in each of the following cases:*
   - (1) $a = (N, M-1, s)$, $b = (0, -1, N)$, $S-N < s \leqslant S$, $s \neq kN$, *where* $k \in Z$.
   - (2) $a = (N, M, 0)$, $b = (N, 0, S)$).

(c) *If n and m are the relatively prime nonnegative integers and* $(n, m, 0) \in Q$, *then*

$$Q \cap (P((0,-M,S),\quad (n,m,0))) = (Q_{-1}\cup Q_0\cup Q_1)\cap Q, \tag{24}$$

where $P((0, -M, S), (n, m, 0))$ *is the plane passing through* $(0, 0, 0)$, $(0, -M, S)$, $(n, m, 0)$ *and* $Q_k = \{l(n, m, 0) + k(0, -M, S) : l \in \mathbb{Z}\}$ *for* $k = -1, 0, 1$.

Proof.

(a) The conditions of condition 1 on N, M, S and the conditions of this proposition on s, p, q, m imply the inequality $\langle a+b, a-b\rangle \neq 0$. Now, by definition 2, we need to show that (17) holds. Let $c = (n_1, m_1, s_1)$ be any vector satisfying

$$\{a+c, a-c\} \subset (P(a,b)\cap Q)\backslash a\mathbb{R}. \tag{25}$$

Since, in all of the above cases, the first coordinate of a is N, the implication (25) and the definition of $Q(N, M, S)$ imply that $n_1 = 0$ for all cases (1)–(3). Hence

$$c \in \{(x_1,x_2,x_3)\in\mathbb{R}^3 : x_1 = 0\} =: \{x_1 = 0\}. \tag{26}$$

On the other hand, it follows from (25) that $c \in P(a, b)$. Thus, c belongs to the line intersection of the planes $P(a, b)$ and $\{x_1 = 0\}$. Since b also belongs to this line and b is a visible element of Γ, we have $c = kb$ for some nonzero integer k. Clearly, if k is not ± 1, then either $a+c$ or $a-c$ does not belong to $Q(N, M, S)$, which means that (17) holds.

(b) First let us consider case (1). It is clear that $\langle b, a-b\rangle = N(s-N) - M \neq 0$, since N and M are the distinct prime numbers. Therefore, we need to prove that (16) holds (see definition 2). Let $c = (n_1, m_1, s_1)$ be any vector satisfying

$$\{c, a-c\} \subset (P(a,b)\cap Q)\backslash a\mathbb{R}. \tag{27}$$

If the vector c lies on the plane $P(a, b)$, then the determinant of the matrix with rows a, b and c is zero. It gives the equality

$$N(s_1 + m_1 N) = n_1(s + (M - 1)N). \tag{28}$$

Since N is a prime number and $s + (M - 1)N$ is not a multiple of N, we have $n_1 = kN$ for $k \in \mathbb{Z}$. Then $c = (kN, m_1, s_1)$. The set $Q(N, M, S)$ contains the vector c only in the following there cases: $k = 0, k = 1, k = -1$. In the case $k = 0$ we have $n_1 = 0$. Then from (28) one observes that $s_1 = -Nm_1$, that is, $c = m_1(0, 1, -N)$, where $m_1 \neq 0$. If $m_1 \neq -1$, i.e., $c \neq b$, then the conditions $S - N < s \leqslant S$ of the proposition imply that $a - c \notin Q$. Thus, in the case $k = 0$, we obtain that $c = b$. If $k = -1$, then one can readily see that $c = (-N, m_1, s_1), a - c \notin Q$. It remains to consider the case $k = 1$, that is, $n_1 = N$. In this case, we use the following obvious implication:

$$a \in \{x_k = n\}, \qquad b \in \{x_k = 0\} \Rightarrow P(a, b) \cap \{x_k = 0\} = b\mathbb{R},$$
$$P(a, b) \cap \{x_k = n\} = a + b\mathbb{R}. \tag{29}$$

Since

$$c = (N, m_1, s_1) \in \{x_1 = N\}, \qquad a \in \{x_1 = N\},$$
$$b \in \{x_1 = 0\}, \qquad c \in P(a, b)\backslash a\mathbb{R}$$

(see (27)), the relation (29) yields that $c \in a + b\mathbb{R}$. Moreover, $c = a + kb$ for some nonzero integer k, since b is the visible element of Γ. Using this and taking into account that $a + kb$, where $a = (N, M - 1, s), b = (0, -1, N)$, lies in Q if and only if $k = -1$, we obtain $c = a - b$.

Now consider case (2). First, let us prove that in this case the plane $P(a, b)$ contains only the vectors $\pm(N, M, 0), \pm(N, 0, S), \pm(0, M, -S)$ of Q. In fact, every element (n, m, s) of this plane satisfies the equation

$$S(nM - mN) = sNM. \tag{30}$$

First, let us consider the case $s = 0$, i.e., the case $nM = mN$. Since N and M are the distinct prime numbers and $-N \leqslant n \leqslant N, -M \leqslant m \leqslant M$, it follows that either $n = \pm N, m = \pm M$ or $n = m = 0$. Now consider the case $s \neq 0$. Then the right-hand side of (30) is a multiple of S. Therefore taking into account that S is a prime number satisfying condition 1 and $-S \leqslant s \leqslant S$, we have $s = \pm S$. This together with (30) gives the relation $(n \pm N)M = mN$. From this relation, one observes that either $n = \mp N, m = 0$ or $n = 0, m = \pm M$. Thus,

$$P(a, b) \cap Q = \{\pm(N, M, 0), \pm(N, 0, S), \pm(0, M, -S)\}.$$

Using this and taking into account that $a = (N, M, 0), b = (N, 0, S)$, we obtain

$$(P(a, b) \cap Q)\backslash a\mathbb{R} = \{\pm b, \pm(a - b)\}. \tag{31}$$

Now suppose that c is a vector satisfying (27). If $c = -b$, then

$$a - c = a + b \notin (P(a, b) \cap Q)\backslash a\mathbb{R}$$

due to (31). Similarly, if $c = -(a - b)$, then

$$a - c = 2a - b \notin (P(a, b) \cap Q)\backslash a\mathbb{R}$$

again due to (31). Therefore (27) and (31) imply the proof of (16).

(c) The relation $(n_1, m_1, s_1) \in P((0, -M, S), (n, m, 0))$ holds if and only if

$$S(mn_1 - m_1 n) = s_1 Mn.$$

If $(mn_1 - m_1 n) = 0$, then $s_1 = 0$. If $(mn_1 - m_1 n) \neq 0$, then $s_1 = \pm S$, since S is prime satisfying condition 1, and $(n, m, 0) \in Q$. Hence, (n_1, m_1, s_1) belongs either to $\{x_3 = 0\}$ or to $\{x_3 = S\}$ or to $\{x_3 = -S\}$. Therefore (24) follows from (29). □

3. Finding $z(a)$ when $a \in B(N,M,S)$

First, we prove the following simple theorem.

Theorem 2. *There exists a unique value of $\tau \in F$ such that the conditions (14), (15) hold.*

Proof. It follows from (7) and from the definition of F that

$$\alpha_\tau(a) = \langle a, \tau\rangle + \alpha(a), \qquad \tau = c_1\omega_1 + c_2\omega_2 + c_3\omega_3, \tag{32}$$

where $\alpha_\tau(a)$ is as defined in (15), and

$$\alpha(a) = \alpha_0(a) = \arg(q(x), \mathrm{e}^{\mathrm{i}\langle a,x\rangle}), \qquad c_k \in [0, 1), \qquad k = 1, 2, 3.$$

Using this, one observes that (14) is equivalent to the following system of equations:

$$2\pi((N-1)c_1 + Mc_2 + Sc_3) = -\alpha(N-1, M, S)(\mathrm{mod}2\pi),$$
$$2\pi(Nc_1 + (M-1)c_2 + Sc_3) = -\alpha(N, M-1, S)(\mathrm{mod}2\pi),$$
$$2\pi(Nc_1 + Mc_2 + (S-1)c_3) = -\alpha(N, M, S-1)(\mathrm{mod}2\pi).$$

The determinant of the coefficient matrix of this system with respect to the unknowns c_1, c_2, c_3 is $8\pi^3(N + M + S - 1)$. Therefore this system has a solution. Let c_1, c_2, c_3 and c_1', c_2', c_3' be different solutions of this system corresponding to the different values of the right-hand side. Introduce the unknowns $x = c_1 - c_1'$, $y = c_2 - c_2'$, $z = c_3 - c_3'$. It is clear that x, y, z are the solution of the system

$$(N-1)x + My + Sz = k, \qquad Nx + (M-1)y + Sz = m, \qquad Nx + My + (S-1)z = n,$$

where k, m, n are integers. The solutions of this system has the form

$$x = \frac{f(k, m, s)}{N + M + S - 1}, \qquad x = \frac{g(k, m, s)}{N + M + S - 1}, \qquad x = \frac{h(k, m, s)}{N + M + S - 1},$$

where $f(k, m, s), g(k, m, s), h(k, m, s)$ are integers and $f(1, 1, 1) = g(1, 1, 1) = h(1, 1, 1) = 1$. Therefore, the above system of equations with respect to the unknowns $c_1, c_2, c_3 \in [0, 1)$ has $N + M + S - 1$ solutions $(c_{1,l}, c_{2,l}, c_{3,l})$ satisfying

$$c_{j,l+1} - c_{j,l} = \frac{1}{N + M + S - 1}, \qquad j = 1, 2, 3 \quad \text{and} \quad l = 1, 2, \ldots, \quad N + M + S - 2.$$

Thus using (32), the equality $\langle\omega_i, \gamma_j\rangle = 2\pi\delta_{i,j}$ and taking into account the notations $z(n\gamma_1 + m\gamma_2 + s\gamma_3) =: z(n, m, s)$, $z(a) = r(a)\mathrm{e}^{\mathrm{i}\alpha(a)}$, $\alpha(a) \in [-\pi, \pi)$, one observes that there exist $\tau_1, \tau_2, \ldots, \tau_{N+M+S-1}$ such that

$$\tau_{l+1} - \tau_l = \frac{\omega_1 + \omega_2 + \omega_3}{N + M + S - 1}, \qquad \alpha_{\tau_{l+1}}(N, M, S) - \alpha_{\tau_l}(N, M, S) = \frac{2\pi}{N + M + S - 1}.$$

This implies that there exists a unique value of τ satisfying (14), (15). □

By theorem 2, without loss of generality, it can be assumed that

$$\alpha(N-1, M, S) = \alpha(N, M-1, S) = \alpha(N, M, S-1) = 0. \tag{33}$$

On the other hand, the invariant (9) determines the modulus of

$$z(N-1, M, S), z(N, M-1, S), z(N, M, S-1), \tag{34}$$

since the vectors $(N-1, M, S)$, $(N, M-1, S)$, $(N, M, S-1)$ are the long visible elements of $Q(N, M, S)$. Therefore, the Fourier coefficients in (34) are known.

In this section, using theorem 1, proposition 1 and taking into account that the Fourier coefficients in (34) are known, we find all the Fourier coefficients $z(a)$ for $a \in B(N, M, S)$, where $B(N, M, S)$ is as defined in section 1. To formulate these results, we use the following remark.

Remark 1. Let $a_1, a_2, \ldots, a_n$ be nonzero elements of Γ. Assign to every polynomial

$$\sum_{k=1,2,\ldots,n} z(a_k)\mathrm{e}^{\mathrm{i}\langle a_k, x\rangle} \tag{35}$$

the vector $(x(a_1), y(a_1), x(a_2), y(a_2), \ldots, x(a_n), y(a_n))$ of $\mathbb{R}^{2n}$, where $x(a_k)$ and $y(a_k)$ are the real and imaginary parts of the Fourier coefficient $z(a_k)$. There exists one-to-one correspondence between the polynomials of the form (35) and elements of $\mathbb{R}^{2n}$. Farther, we assume the following types of conditions on the Fourier coefficients.

Type 1. Assume that $z(a_j) \neq 0$ for some index j. In other words, we eliminate the finite number of subspaces $z(a_j) = 0$ of dimension $2n - 2$.

Type 2. Assume that some linear combinations of the invariants $e(i, a, b)$ defined in (12), (13) are not $0(\bmod \pi)$.

Type 3. Assume that some homogenous polynomials depending on
$x(a_1), y(a_1), x(a_2), y(a_2), \ldots$ are not zero.

These conditions mean that we eliminate some sets of dimensions less than $2n$. In any case, the $2n$-dimensional measures of the eliminated sets are zero. We named these conditions as zero measure conditions. This means that we consider almost all polynomials of the form (35). In order to avoid eclipsing the essence by technical details, we prefer to formulate the theorems for almost all the potentials of the form (35) instead of listing the eliminated sets. Note that the separated potentials show that, to determine the potential uniquely (modulo inversion and translation) from spectral invariants, it is necessary to eliminate some of these subspaces. Thus, the sufficient conditions to solve the inverse problem by these methods are close to the necessary conditions.

First, let us consider

$$z^2(N, M-1, l), \qquad z(N, M, l), \qquad z(N, M-2, l), \qquad \forall l. \tag{36}$$

Theorem 3. *Suppose condition 1 holds. Then the spectral invariants (9)–(11) determine constructively and uniquely, modulo inversion and translation (6), the numbers in (36) for almost all the potentials of the form (7).*

Proof. Since the vectors (N, M, l), $(N, M-1, l)$, $(N, M-2, l)$ are long visible, the absolute values of the numbers in (36) are known. Therefore, we need to find

$$2\alpha(N, M-1, l), \qquad \alpha(N, M, l), \qquad \alpha(N, M-2, l), \qquad \forall l. \tag{37}$$

To find (36) for $l = S, S-1, S-2$, we use equation (13) for the following pairs:

$P_1 = \{(N, M, S-1), (0, 0, 1)\}$, $P_2 = \{(N, M-1, S), (0, 1, 0)\}$,
$P_3 = \{(N, M-1, S-1), (0, 1, -1)\}$, $P_4 = \{(N, M-1, S-1), (0, 1, 0)\}$,
$P_5 = \{(N, M-1, S-1), (0, 0, 1)\}$, $P_6 = \{(N, M-1, S-1), (0, 1, 1)\}$,
$P_7 = \{(N, M-2, S-1), (0, 0, 1)\}$ and $P_8 = \{(N, M-1, S-2), (0, 1, 0)\}$.

Note that it follows from proposition 1(a) that the pairs $P_1, P_2, \ldots, P_8$ are the canonical pairs of type 2. Therefore, by theorem 1, we have the invariant $A_2(a, b)$ and hence the equation of type (13) corresponds to each of the pairs $P_1, P_2, \ldots, P_8$. For simplicity of the notation, in (13) for P_i, instead of $e(2, a, b)$ and $d(a, b)$ we write e_i and d_i respectively. Denote

$\alpha(N, M, S)$ by α. Using this notation and (33) one observes that the equality (13) for the pairs $P_1, P_2, \ldots, P_8$ has the form

$$\begin{aligned}
&\alpha + \alpha(N, M, S-2) = d_1e_1(\mathrm{mod}2\pi),\\
&\alpha + \alpha(N, M-2, S) = d_2e_2(\mathrm{mod}2\pi),\\
&-2\alpha(N, M-1, S-1) + \alpha(N, M, S-2) + \alpha(N, M-2, S) = d_3e_3(\mathrm{mod}2\pi),\\
&-2\alpha(N, M-1, S-1) + \alpha(N, M-2, S-1) = d_4e_4(\mathrm{mod}2\pi),\\
&-2\alpha(N, M-1, S-1) + \alpha(N, M-1, S-2) = d_5e_5(\mathrm{mod}2\pi),\\
&-2\alpha(N, M-1, S-1) + \alpha + \alpha(N, M-2, S-2) = d_6e_6(\mathrm{mod}2\pi),\\
&-2\alpha(N, M-2, S-1) + \alpha(N, M-2, S) + \alpha(N, M-2, S-2) = d_7e_7(\mathrm{mod}2\pi),\\
&-2\alpha(N, M-1, S-2) + \alpha(N, M, S-2) + \alpha(N, M-2, S-2)) = d_8e_8(\mathrm{mod}2\pi).
\end{aligned} \tag{38}$$

From the first and second equations of (38) we obtain

$$\alpha(N, M, S-2) = (d_1e_1 - \alpha)(\mathrm{mod}2\pi), \quad \alpha(N, M-2, S) = (d_2e_2 - \alpha)(\mathrm{mod}2\pi). \tag{39}$$

These equalities with the third equation of (38) yield

$$-2\alpha(N, M-1, S-1) = (d_3e_3 - d_2e_2 - d_1e_1 + 2\alpha)(\mathrm{mod}2\pi).$$

Now using the last equality in the fourth, fifth and sixth equations of (38), we get

$$\begin{aligned}
\alpha(N, M-2, S-1) &= (d_4e_4 + d_2e_2 + d_1e_1 - d_3e_3 - 2\alpha)(\mathrm{mod}2\pi),\\
\alpha(N, M-1, S-2) &= (d_5e_5 + d_2e_2 + d_1e_1 - d_3e_3 - 2\alpha)(\mathrm{mod}2\pi),\\
\alpha(N, M-2, S-2) &= (d_6e_6 + d_2e_2 + d_1e_1 - d_3e_3 - 3\alpha)(\mathrm{mod}2\pi).
\end{aligned} \tag{40}$$

Writing the obtained value for $\alpha(N, M-2, S-1), \alpha(N, M-2, S), \alpha(N, M-2, S-2), \alpha(N, M-1, S-2)$ into the seventh and eighth equations of (38) we obtain

$$\begin{aligned}
&d_7e_7 - (d_6e_6 - 2d_4e_4 + d_3e_3 - d_1e_1) = 0(\mathrm{mod}2\pi),\\
&\quad d_8e_8 + 2d_5e_5 + d_2e_2 = d_6e_6 + d_3e_3(\mathrm{mod}2\pi).
\end{aligned} \tag{41}$$

Introduce the notations $V = (d_1, d_3, d_4, d_6, d_7)$, $U = (d_8, d_5, d_2)$,

$$f_1(V) \equiv d_7e_7 - (d_6e_6 - 2d_4e_4 + d_3e_3 - d_1e_1), \qquad f_2(U) \equiv d_8e_8 + 2d_5e_5 + d_2e_2.$$

In these notations, (41) has the form

$$f_1(V) = 0(\mathrm{mod}2\pi), \qquad f_2(U) = d_6e_6 + d_3e_3(\mathrm{mod}2\pi). \tag{42}$$

Since d_i is either 1 or -1, the vector V takes 32 distinct values

$$V_1, V_2, \ldots, V_{16} \quad \text{and} \quad -V_1, -V_2, \ldots, -V_{16}.$$

Then the function $f_1(V)$ takes 32 values

$$f_1(V_1), f_1(V_2), \ldots, f_1(V_{16}) \quad \text{and} \quad f_1(-V_1), f_1(-V_2), \ldots, f_1(-V_{16}).$$

Similarly, the vector U takes eight distinct values $U_1, U_2, \ldots, U_8$, and the function $f_2(U)$ takes eight values $f_2(U_1), f_2(U_2), \ldots, f_2(U_8)$. Suppose

$$f_1(V_k) - f_1(Vj) \neq 0(\mathrm{mod}2\pi)$$

for $k \neq j$. Then there are only one index k and two values $V_k, -V_k$ of V satisfying

$$f_1(V_k) = -f_1(-V_k) = 0(\mathrm{mod}2\pi).$$

On the other hand, the arguments of the Fourier coefficients of $q(x)$ and $q(-x)$ take the opposite values. Therefore, for fixing the inversion $q(x) \longrightarrow q(-x)$, we take one of these two remaining values $V_k, -V_k$ of V. Thus, one can find the signs of d_1, d_3, d_4, d_6, d_7 from the first equality in (42). Since the signs of d_3 and d_6 are already known, we find d_8, d_5, d_2 from the second equality in (42) if

$$d_6e_6 + d_3e_3 \neq 0(\mathrm{mod}2\pi) \quad \text{and} \quad f_2(U_k) - f_2(U_j) \neq 0(\mathrm{mod}2\pi).$$

Thus, the numbers $d_1, d_2, \ldots, d_8$ are known. Since $e_1, e_2, \ldots, e_8$ are known invariants, the numbers in (37) for $l = S, S-1, S-2$ can be expressed in terms of α. Moreover, we have the formulae (see (39), (40))

$$\begin{aligned} -2\alpha(N, M-1, S-p) &= E_1 + 2p\alpha, \\ \alpha(N, M, S-p) &= E_2 - (p-1)\alpha, \\ \alpha(N, M-2, S-p) &= E_3 - (p+1)\alpha \end{aligned} \tag{43}$$

for $p = 0, 1, 2$, where by E_i for $i = 1, 2, \ldots$ we denote the linear combinations of $e_1, e_2, \ldots$ with known coefficients.

Now let us consider (37) for all l. For this, we use equation (13) for the canonical pairs $P_9(s) = \{(N, M-1, s), (0, 1, 1)\}$, $P_{10}(s) = \{(N, M-1, s-1), (0, 1, 0)\}$,
$P_{11}(s) = \{(N, M-1, s), (0, 1, -1)\}$ of type 2 (see proposition 1(a)). Equations (13) for these pairs are

$$\begin{aligned} &-2\alpha(N, M-1, s) + \alpha(N, M, s+1) + \alpha(N, M-2, s-1) = d_9(s)e_9(s)(\mathrm{mod}2\pi), \\ &-2\alpha(N, M-1, s-1) + \alpha(N, M, s-1) + \alpha(N, M-2, s-1) = d_{10}(s)e_{10}(s)(\mathrm{mod}2\pi), \\ &-2\alpha(N, M-1, s) + \alpha(N, M, s-1) + \alpha(N, M-2, s+1) = d_{11}(s)e_{11}(s)(\mathrm{mod}2\pi), \end{aligned} \tag{44}$$

where $d_9(s), d_{10}(s), d_{11}(s)$ are either 1 or -1. Using the equations

$$\begin{aligned} &-2\alpha(N, M-1, s) + \alpha(N, M, s+2) + \alpha(N, M-2, s-2) = d_{12}e_{12}(\mathrm{mod}2\pi), \\ &-2\alpha(N, M-1, s) + \alpha(N, M, s-2) + \alpha(N, M-2, s+2) = d_{13}e_{13}(\mathrm{mod}2\pi), \end{aligned}$$

which are equation (13) for the pairs $\{(N, M-1, s), (0, 1, 2)\}, \{(N, M-1, s), (0, 1, -2)\}$, and arguing as in the determinations of the signs of d_8, d_5, d_2, one can find the signs of $d_9(s), d_{10}(s), d_{11}(s)$. Then from equations (44), we can find (37) for $l = s-1$ if (37) is known for $l = s+1, s$. Moreover, as we proved above, they satisfy the formulae (43) for $p = 0, 1, 2$. The formulae in (43) for all p can easily be obtained from (44) by induction. In the same way, we obtain the formulae

$$\alpha(N, M-p, S) = E_4 - (p-1)\alpha, \qquad \alpha(0, M, -S) = E_5 - (2S+N-1)\alpha. \tag{45}$$

By proposition 1(b), the pair $\{(N, M, 0), (N, 0, S)\}$ is a canonical pair of type 1. Hence, using the invariant $A_1(a, b)$ (see (10)) for $a = (N, M, 0), b = (N, 0, S)$ and formulae (43), (45), we get the value of

$$\cos((N + M + S - 1)\alpha + E_6).$$

Similarly, using the pair $\{(N, M, 0), (N, 0, -S)\}$, we find

$$\cos((N + M + S - 1)\alpha + E_7).$$

By these two values of the cosine, we find $(N+M+S-1)\alpha$ under condition $E_6 \neq E_7(\mathrm{mod}\pi)$. This with (15) gives us the unique value of α, and we find the numbers in (37) under some zero measure conditions in the sense of remark 1. □

To find the Fourier coefficient $z(a)$ for all $a \in \partial\widetilde{Q}$, where $\partial\widetilde{Q}$ is as defined in section 1, we use the following lemmas.

Lemma 1. *Let* $\{a_1, b\}$ *and* $\{a_2, b\}$, *where* a_1 *and* a_2 *are the long visible elements of* $Q(N, M, S)$, *be the canonical pairs of type 1. Then the invariants*

$$S_1(a_1, b) = \operatorname{Re}(z(-a_1)z(a_1 - b)z(b)), \qquad S_1(a_2, b) = \operatorname{Re}(z(-a_2)z(a_2 - b)z(b)), \tag{46}$$

defined in (10), uniquely determine $z(b)$ *if* $z(a_k - b)$ *and* $z(a_k)$ *for* $k = 1, 2$ *are known and*

$$\operatorname{Im}(z(a_1 - b)z(a_1)z(-(a_2 - b))z(-a_2)) \neq 0. \tag{47}$$

Proof. The equations in (46) are a system of the linear equations with respect to the unknowns $x(b)$, $y(b)$, and the inequality (47) shows that the determinant of the coefficient matrix of this system is not zero. Therefore (46) has a unique solution. □

Lemma 2. *Suppose* $c \in Q$ *has two different decompositions*

$$c = a_1 + b_1, \qquad c = a_2 + b_2, \qquad \text{where} \quad \{a_1, b_1, a_2, b_2\} \subset Q_N,$$

such that $z^2(a_k)$ *and* $z(a_k - b_k)$ *for* $k = 1, 2$ *are known and*

$$\operatorname{Im}(z^2(a_1)z(a_1 - b_1)z^2(-(a_2))z(-(a_2 - b_2))) \neq 0. \tag{48}$$

If $\{a_1, b_1\}$ *and* $\{a_2, b_2\}$, *where* a_1 *and* a_2 *are the long visible elements of* $Q(N, M, S)$, *are the canonical pairs of type 2, then the invariants*

$$S_2(a_k, b_k) = \operatorname{Re}(z^2(-a_k)z(a_k - b_k)z(a_k + b_k)),$$

defined by (11), where $k = 1, 2$, *uniquely determine* $z(c)$.

The proof is the same as the proof of lemma 1.

Theorem 4. *Suppose that condition 1 holds. Then the spectral invariants (9)–(11) and (19) determine constructively and uniquely, modulo inversion and translation (6), the Fourier coefficients* $z(a)$ *for all* $a \in \partial\widetilde{Q}$ *for almost all the potentials of the form (7).*

Proof. *Step 1.* In this step, we find the Fourier coefficient $z(N, M - 1, s)$ for $s = S - 2p$. Since $z^2(N, M - 1, s)$ is known due to theorem 3, $z(N, M - 1, s)$ is known up to the sign,

$$z(N, M - 1, s) = k_s v_s,$$

where v_s is known and k_s is either 1 or -1. Moreover, k_S is known (see (33)). To find k_s for $s = S - 2p$, where $p = 1, 2, \dots, S - 1$ we use the invariant (19) for the pair $\{a, b\}$, where $a = (N, M - 1, S - p)$, $b = (0, 0, 1)$. To write the invariant (19) for this pair, we need to determine the set G_2, defined in theorem 1, for this pair. By definition, G_2 is the set of all c such that

$$\{a + c, a - c\} \subset (P(a, b) \cap Q)\backslash a\mathbb{R}.$$

Clearly, if this inclusion holds, then c has the form $(0, m, s)$. Hence, c belongs to the line intersection of the planes $P(a, b)$ and $\{x_1 = 0\}$. By (29) this line is $b\mathbb{R}$. It means that $c = (0, 0, q)$ for some integer q. Thus, G_2 is the set of all $(0, 0, q)$ such that

$$\{(N, M - 1, S - p - q), (N, M - 1, S - p + q)\} \subset Q.$$

This inclusion implies that $-p \leqslant q \leqslant p$. Therefore the invariant (19) for the pair $\{(N, M - 1, S - p), (0, 0, 1)\}$ has the form

$$\operatorname{Re}\left(z^2(N, M - 1, S - p)\sum_{q=-p}^{p}\frac{\langle (N, M - 1, S - p - q), (N, M - 1, S - p + q)\rangle}{\langle (N, M - 1, S - p), \beta\rangle}h_q V_q\right), \tag{49}$$

where $V_q =: v_{S-p+q}v_{S-p-q}$ is the known number and $h_q =: k_{S-p+q}k_{S-p-q}$ is either 1 or -1. Let

$$H = (h_{-p}, h_{-p+1}, \ldots, h_p)$$

and f be a function taking H to (49). Assume that f takes distinct nonzero values at distinct points. Then (49) determine

$$h_q = k_{S-p+q}k_{S-p-q} \tag{50}$$

if $\langle (N, M-1, S-p-q), (N, M-1, S-p+q)\rangle \neq 0$. Thus (50) is known. Taking $p = q$ in (50), we find $k_S k_{S-2p}$. Since k_S is known (see (33)), we find k_{S-2p} if

$$A(p) =: \langle (N, M-1, S-2p), (N, M-1, S)\rangle \neq 0.$$

Since the equation $A(p) = 0$ may have only one integer root p_0, we have defined k_{S-2p} for all p except $p = p_0$. It is clear that there exist p and q such that $p_0 = p + q$ and $\langle (N, M-1, S-p-q), (N, M-1, S-p+q)\rangle \neq 0$. Therefore, using (50), we define k_{S-p_0}, since k_{S-p-q} is known.

Step 2. To find $z(N, M-1, S-2p+1)$, we use lemma 1. Let

$$a_1 = (N, M-1, S), \qquad a_2 = (N, M-1, S-2), \qquad b = (0, -1, N).$$

Without loss of generality, it can be assumed that $S - 2 \neq kN$. Otherwise, we consider $a_2 = (N, M-1, S-4)$. By proposition 1(b), the pairs $\{a_1, b\}$ and $\{a_2, b\}$ are the canonical pairs of type 1. Therefore applying lemma 1 and taking into account that $z(a_k - b)$ and $z(a_k)$ for $k = 1, 2$ are known due to theorem 3 and step 1, we find $z(b)$. Now, without loss of generality, we assume that $S - 1 \neq kN$. Otherwise we consider $S - 3$ instead of $S - 1$. By proposition 1(b), the pair $\{a, b\}$, where $a = (N, M-1, S-1)$ and $b = (0, -1, N)$, is the canonical pair of type 1. Hence using the invariant (10) and taking into account that $z(b)$ and $z(a - b)$ are known, we determine the sign of k_{S-1}. From the knowledge of the sign of k_S, we have found the sign of k_{S-2p} by (49). In the same way, from the knowledge of the sign of k_{S-1}, we find the sign of k_{S-2p-1}. Thus, we have found $z(N, M-1, s)$ for all s.

Step 3. Now using lemma 2, we find $z(N, m, s)$ for all m, s by induction. They were found in theorem 3 and in steps 1 and 2 of this theorem for $m = M, M-1, M-2$. Let us find $z(N, m, s)$ assuming that we have already found $z(N, q, s)$ for $q = M, M-1, \ldots, m+1$. Clearly, for any $s \in [S, -S]$ there are different pairs $(s_1, p_1), (s_2, p_2)$ such that

$$s_k + p_k = s; \qquad s_k, p_k, s_k - p_k \in [S, -S]; \qquad s_k - 2p_k \notin [-S, S];$$
$$N^2 + m^2 - 1 + s_k^2 - p_k^2 \neq 0$$

$s_k \neq \pm N, s_k - p_k \neq \pm N$ for $k = 1, 2$. Then, by proposition 1(a) (see case 3), the pair $\{a_k, b_k)\}$ for $k = 1, 2$, where $a_k = (N, m+1, s_k)$ and $b_k = (0, -1, p_k)$, is the canonical pair of type 2. Moreover, $z(a_k), z(a_k - b_k)$ are known by the assumption of the induction. Hence the application of lemma 2 yields $z(N, m, s)$. Interchanging the roles of the first and second coordinates and then the roles of the first and third coordinates, we find $z(a)$ for all $a \in \partial\widetilde{Q}$ under some zero measure conditions in the sense of remark 1. □

Theorem 5. *Suppose condition 1 holds. Then the spectral invariants (9)–(11), (18) determine constructively and uniquely, modulo inversion and translation (6), the Fourier coefficients*

$$z(n, m, 0), \qquad z(n, 0, s), \qquad z(0, m, s)$$

for all n, m, s and for almost all the potentials of the form (7).

Proof. Let us find $z(n, m, 0)$. Since $(n, m, 0) \neq (0, 0, 0)$ and $z(-a) = \overline{z(a)}$, without loss of generality, it can be assumed that $m > 0, n \geqslant 0$. Moreover, for the simplicity of the notations, it can be assumed that n, m are relatively prime numbers, since we find $z(l(n, m, 0))$ for all l. To find $z(n, m, 0)$, we use the invariant (18) for the pair $\{a_q, (n, m, 0)\}$, where $a_q = (0, -M, S) + q(n, m, 0)$. To write the invariant (18) for this pair, we need to investigate the set G_1, defined in theorem 1, for this pair. By the definition, G_1 is the set of all c such that

$$\{c, a_q - c\} \subset (P(a_q, (n, m, 0)) \cap Q)\backslash a_q\mathbb{R}.$$

Using this, the obvious equality $P(a_q, (n, m, 0)) = P((0, -M, S), (n, m, 0))$ and (24), we obtain that G_1 is the set of all c such that

$$\{c, a_q - c\} \subset ((Q_{-1} \cup Q_0 \cup Q_1) \cap Q)\backslash a_q\mathbb{R}.$$

If $c \in Q_{-1}$ then

$$a_q - c = (q - l)(n, m, 0) + (0, -2M, 2S) \notin Q.$$

If $c \in Q_0$, then $c = l(n, m, 0)$ for some l. Let p be the greatest integer satisfying $pn \leqslant N$, $pm \leqslant M$. Then $l(n, m, 0) \in Q$ if and only if $-p \leqslant l \leqslant p$. Moreover

$$a_q - c = (0, -M, S) + (q - l)(n, m, 0) \in Q_1.$$

Similarly, if $c \in Q_1$, i.e., $c = (0, -M, S)+(q-l)(n, m, 0)$ for some l, then $a_q - c = l(n, m, 0)$. Therefore, the invariant (18) for the pair $\{a_q, (n, m, 0)\}$ has the form

$$\mathrm{Re} z(-(a_q)) \sum_l c_l z(a_q - l(n, m, 0)) z(l(n, m, 0)), \tag{51}$$

where $q = 1, 2, \ldots, p$ and $c_l = g(a_q, l(n, m, 0))$. Similarly, the invariant (18) for the pair $\{b_q, (n, m, 0)\}$, where $b_q = (0, M, S) + q(n, m, 0)$, has the form

$$\mathrm{Re} z(-(b_q)) \sum_l d_l z(b_q - l(n, m, 0)) z(l(n, m, 0)), \tag{52}$$

where $q = -1, -2, \ldots, -p$ and $d_l = g(b_q, l(n, m, 0))$. Since the Fourier coefficients

$$z(a_q), \qquad z(a_q - l(n, m, 0)), \qquad z(b_q), \qquad z(b_q - l(n, m, 0))$$

are known due to theorem 4, we have $2p$ linear form (see (51) and (52)) with respect to $2p$ unknowns $x(n, m, 0), x(2(n, m, 0)), \ldots, x(p(n, m, 0))$ and $y(n, m, 0), y(2(n, m, 0)), \ldots, x(q(n, m, 0))$. Since the invariant (18) is a known number, (51) and (52) give $2p$ linear equations with respect to these unknowns. One can find these unknowns if the determinant $T(2p)$ of the coefficient matrix of the system of these linear equations is not zero. Let us show that this determinant is not identically zero. Let $x(l(n, m, 0))$ be the lth and $y(l(n, m, 0))$ be the $(p + l)$th unknown of the system, where $l = 1, 2, \ldots, p$. Similarly, let the lth equation of the system be given by the lth linear form of (51) and the $(p + l)$th equation of the system be given by the lth linear form of (52). Then $T(2p)$ can be written in the form

$$\begin{vmatrix} a_{1,1} & a_{1,2} & \ldots & a_{1,p} & b_{1,1} & b_{1,2} & \ldots & b_{1,p} \\ a_{2,1} & a_{2,2} & \ldots & a_{2,p} & b_{2,1} & b_{1,2} & \ldots & b_{2,p} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{p,1} & a_{p,2} & \ldots & a_{p,p} & b_{p,1} & b_{p,2} & \ldots & b_{p,p} \\ c_{1,1} & c_{1,2} & \ldots & c_{1,p} & d_{1,1} & d_{1,2} & \ldots & d_{1,p} \\ c_{2,1} & c_{2,2} & \ldots & c_{2,p} & d_{2,1} & d_{2,2} & \ldots & d_{2,p} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ c_{p,1} & c_{p,2} & \ldots & c_{p,p} & d_{p,1} & d_{p,2} & \ldots & d_{p,p} \end{vmatrix}, \qquad \text{where}$$

$$a_{q,l} = x(a_q)(c_l x(a_{q-l}) + c_{-l} x(a_{q+l})) + y(a_q)(c_l y(a_{q-l}) + c_{-l} y(a_{q+l})),$$

$$b_{q,l} = x(a_q)(c_l y(a_{q-l}) - c_{-l} y(a_{q+l})) + y(a_q)(c_l x(a_{q-l}) - c_{-l} x(a_{q+l})),$$

$$c_{q,l} = x(b_{-q})(d_l x(b_{-q-l}) + d_{-l} x(b_{-q+l})) + y(b_{-q})(d_l y(b_{-q-l}) + d_{-l} y(b_{-q+l})),$$

$$d_{q,l} = x(b_{-q})(d_l y(b_{-q-l}) - d_{-l} y(b_{-q+l})) + y(b_{-q})(d_l x(b_{-q-l}) - d_{-l} x(b_{-q+l})).$$

The qth and $(p+q)$th diagonal elements $a_{q,q}$ and $d_{q,q}$ of the determinant contain the summands $x(a_q)c_q x(a_0)$ and $x(b_{-q})d_{-q}y(b_0)$, respectively. The nondiagonal elements do not contain these summands. Therefore, the determinant $T(2p)$ contain the summand

$$\sqcap_{q=1,2,\ldots,p}(c_q x(a_q)x(a_0)d_{-q}x(b_{-q})y(b_0))$$

which cannot be canceled by the other summand of the determinant. Moreover, the multiplicands c_q and d_{-q} are not zero since

$$\langle q(n,m,0), a_q - q(n,m,0)\rangle = -qmM \neq 0,$$
$$\langle q(n,m,0), b_q - q(n,m,0)\rangle = qmM \neq 0.$$

Therefore, the zero set of the determinant $T(2p)$ of the coefficient matrix of the system has zero measure. Thus, solving this system we find $z(n,m,0)$ under some zero measure conditions in the sense of remark 1. In the same way, we find $z(n,0,s)$ and $z(0,m,s)$. □

4. Inverse problem in a dense set

In this section, we construct a dense in $W_2^s(F)$, where $s > 3$, in $\mathbb{C}^\infty$-topology set D of trigonometric polynomials and prove that one can determine constructively and uniquely (module translation (6)) the potential $q \in D$ from the spectral invariants (2)–(4). For this, we use the following condition.

Condition 2. *Suppose* $z(n,m,s) \neq 0$ *for* $(n,m,s) \in C(\sqrt{N})$, *where*

$$C(\sqrt{N}) = \{(n,m,s) : 0 < |n| < \tfrac{1}{2}\sqrt{N}, 0 < |m| < \tfrac{1}{2}\sqrt{N}, 0 < |s| < \tfrac{1}{2}\sqrt{N}\}$$

and $z(n,m,s) = 0$ *for* $(n,m,s) \in (Q(N,M,S))\backslash(C(\sqrt{N}) \cup B(N,M,S))$.

To find $z(n,m,s)$ for $(n,m,s) \in C(\sqrt{N})$ we use the following proposition.

Proposition 2. *If condition 2 holds, then the invariant (18) for* $a \in B(N,M,S), b \in C(\sqrt{N})$ *yields the invariant*

$$\operatorname{Re}\left(z(-a)\left(\sum_{c\in G} g(a,c)z(a-c)z(c)\right)\right), \tag{53}$$

where $g(a,c) = \frac{\langle c,c-a\rangle}{(\langle c,\beta\rangle)^2}$, *G is the set of all c such that*

$$\{c, a-c\} \subset ((P(a,b)\cap Q)\backslash a\mathbb{R}) \cap (C(\sqrt{N}) \cup B(N,M,S)), \tag{54}$$

and at least one of the points c and $a-c$ *belongs to* $C(\sqrt{N})$.

Proof. By condition 2, if $\{c, a-c\}$ is not a subset of $C(\sqrt{N}) \cup B(N,M,S)$, then $z(a-c)z(c) = 0$. Therefore, it follows from the definition of G_1 that the summation in (18) is taken over all c satisfying (54). On the other hand, if both c and $a-c$ belong to $B(N,M,S)$, then the summand $z(-a)g(a,c)z(a-c)z(c)$ of (18) is known due to theorems 4 and 5. Therefore, (18) implies the invariant (53), if condition 2 holds. □

Theorem 6. *The invariants (9)–(11) and (53) determine constructively and uniquely, modulo inversion and translation (6), the Fourier coefficients* $z(n,m,s)$, *where* $(n,m,s) \in C(\sqrt{N})$, *for almost all the potentials of the form (7) satisfying conditions 1 and 2.*

Proof. To find $z(n,m,s)$ for $(n,m,s) \in C(\sqrt{N})$, we use the invariant (53) for the pair $a = (-N+n, 0, j), b = (n,m,s)$, where j is a prime number satisfying

$$M < j \leqslant S - \sqrt{N}. \tag{55}$$

Since $n \neq 0$ and $z(-n,-m,-s) = \overline{z(n,m,s)}$, without loss of generality, it can be assumed that $n > 0$. To use (53), we prove that

$$G = \{b, a-b\}, \qquad \text{where} \qquad b = (n,m,s), \qquad a-b = (-N,-m,j-s), \tag{56}$$

where G is as defined in proposition 2. Since the inclusion $\{b, a-b\} \subset G$ is obvious, we need to prove that $G \subset \{b, a-b\}$. For this, we use the following inequalities

$$0 < |n|, |m|, |s| < \tfrac{1}{2}\sqrt{N}, \; 2N < M < j \leqslant S - \sqrt{N}, \tag{57}$$

which follows from (55), condition 1 and the assumption $(n,m,s) \in C(\sqrt{N})$. Thus, to prove (56) we need to show that any element $c = (n_1, m_1, s_1)$ of G is either b or $a-b$. First let us prove that $n_1 m_1 s_1 \neq 0$. Indeed, using the definition of $C(\sqrt{N})$ and the inequalities in (57) one can readily verify that the following three statements are true.

1. If $n_1 = 0$, then $(n_1, m_1, s_1) \notin C(\sqrt{N})$, $a - c = (-N+n, -m_1, j-s_1) \notin C(\sqrt{N})$.
2. If $m_1 = 0$, then $(n_1, m_1, s_1) \notin C(\sqrt{N})$, $a - c = (-N+n-n_1, 0, j-s_1) \notin C(\sqrt{N})$.
3. If $s_1 = 0$, then $(n_1, m_1, s_1) \notin C(\sqrt{N})$, $a - c = (-N+n-n_1, -m_1, j) \notin C(\sqrt{N})$.

Therefore, the relation $(n_1, m_1, s_1) \in G$ and the definition of G (see proposition 2) imply that $n_1 m_1 s_1 \neq 0$. Since $c \in G$ we have $c \in P(a,b) \cap Q$. The point $c = (n_1, m_1, s_1)$ belongs to the plane $P(a,b)$ if and only if

$$(n-N)(ms_1 - sm_1) = j(mn_1 - nm_1). \tag{58}$$

This equation holds in the following two cases.

Case 1. $(ms_1 - sm_1) = 0$. Then $(mn_1 - nm_1) = 0$. These two equalities imply that the point $c = (n_1, m_1, s_1)$ lies on the line $(n,m,s)\mathbb{R}$. Therefore we have

$$c = (n_1, m_1, s_1) = k(n_0, m_0, s_0), \qquad (n,m,s) = k_0(n_0, m_0, s_0), \tag{59}$$

where k and k_0 are the integers, and (n_0, m_0, s_0) is a visible element of $\mathbb{Z}^3$ lying in $(n,m,s)\mathbb{R}$. Moreover, it follows from (57) and from the above relation $n_1 m_1 s_1 \neq 0$ that

$$0 < |n_0|, |m_0|, |s_0| < \tfrac{1}{2}\sqrt{N} \quad \text{and} \quad kk_0 \neq 0. \tag{60}$$

Using this let us prove that $k(n_0, m_0, s_0) \in G$ if and only if $k = k_0$. If $k = k_0$, then by (59) we have $(n_1, m_1, s_1) = (n,m,s) = b \in G$. Now we prove that if $k \neq k_0$, then $c = k(n_0, m_0, s_0) \notin G$. Suppose at least one of the inequalities

$$|kn_0| > \tfrac{1}{2}\sqrt{N}, \qquad |km_0| > \tfrac{1}{2}\sqrt{N}, \qquad |ks_0| > \tfrac{1}{2}\sqrt{N} \tag{61}$$

holds. Then using (60), the definitions of $C(\sqrt{N})$ and $B(N,M,S)$, and taking into account that N, M, and S are the prime numbers, we see that

$$c = k(n_0, m_0, s_0) \notin C(\sqrt{N}) \cup B(N,M,S),$$

and hence $c \notin G$. Now suppose that all the inequalities in (61) do not hold. Then using (57), (59), (60) and the assumption $k \neq k_0$, one can easily verify that

$$-N + n - kn_0 \neq 0, \pm N; \quad km_0 \neq 0, \pm M; \quad j - ks_0 \neq 0, \pm S; \quad j - ks_0 > \sqrt{N}.$$

These relations and the definitions of $C(\sqrt{N})$ and $B(N,M,S)$ imply that

$$a - c = a - k(n_0, m_0, s_0) = (-N + n - kn_0, -km_0, j - ks_0) \notin C(\sqrt{N}) \cup B(N,M,S),$$

which means that $c \notin G$ (see the definition of G in proposition 2). Hence, it is proved that if $k \neq k_0$, then $(n_1, m_1, s_1) \notin G$. Thus, in Case 1, the inclusion $c \in G$ implies the equality $c = b$.

Case 2. $(ms_1 - sm_1) \neq 0$. Then it follows from (58) that

$$(ms_1 - sm_1) = pj, \tag{62}$$

where p is a nonzero integer, since j is a prime number satisfying $j > N - n$ (see (57)). The formulae (62) and (58) imply that

$$(n - N)p = mn_1 - nm_1. \tag{63}$$

Using (62) and (57) one can readily verify that at least one of the inequalities

$$|m_1| > \sqrt{N}, \qquad |s_1| > \sqrt{N} \tag{64}$$

holds. If the first inequality of (64) holds, then

$$c = (n_1, m_1, s_1) \notin C(\sqrt{N}), \qquad a - c = (-N + n - n_1, -m_1, j - s_1) \notin C(\sqrt{N})$$

and hence $c \notin G$.

Now assume that $|s_1| > \sqrt{N}$ and $|m_1| \leqslant \sqrt{N}$. Then $c = (n_1, m_1, s_1) \notin C(\sqrt{N})$. Therefore the relation $c \in G$ and the definition of G give

$$a - c = (-N + n - n_1, -m_1, j - s_1) \in C(\sqrt{N}).$$

Using this, the definition of $C(\sqrt{N})$ and (57), we obtain

$$|-N - n_1| < \sqrt{N}, \qquad 0 < |m_1| < \sqrt{N}, \qquad |j - s_1| < \sqrt{N}. \tag{65}$$

Since $c \in G$, we have $c \in C(\sqrt{N}) \cup B(N, M, S)$. On the other hand $c \notin C(\sqrt{N})$. Hence $c = (n_1, m_1, s_1) \in B(N, M, S)$, that is, at least one of the following inclusions holds

$$n_1 \in \{0, N, -N\}, \qquad m_1 \in \{0, M, -M\}, \qquad s_1 \in \{0, S, -S\}.$$

This with (65) and (55) implies that $n_1 = -N$. Using this in (63), we get

$$N(p - m) = n(p + m_1). \tag{66}$$

We assumed that $|m_1| \leqslant \sqrt{N}$. Besides, by (57) we have $|m| \leqslant \sqrt{N}$, $|n| \leqslant \sqrt{N}$. From these inequalities and (66) one can easily conclude that $|p + m_1| < N$. Thus, N is a prime number and is greater than $|n|$ and $|p + m_1|$. Therefore from (66) we obtain that $p + m_1 = 0$, $p - m = 0$, and hence $p = m = -m_1$. Using this in (62), we obtain

$$(ms_1 + sm) = mj, \qquad s_1 = j - s, \qquad c = (n_1, m_1, s_1) = (-N, -m, j - s) = a - b.$$

Thus, we proved that any element c of the set G is either b (see Case 1) or $a - b$. Hence $G \subset \{b, a - b\}$ and (56) is proved.

Now it follows from (56) that the invariant (53) has the form

$$2\mathbf{Re}z(-a)g(a, b)z(a - b)z(b)). \tag{67}$$

Clearly, there exist two numbers j_1 and j_2 such that they satisfy the conditions of j and

$$\langle(-N + n, 0, j_1), \qquad (n, m, s)\rangle \neq 0, \qquad \langle(-N + n, 0, j_1), (n, m, s)\rangle \neq 0,$$

which implies that the multiplicand $g(a, b)$ in (67) for $a = (-N + n, 0, j_i)$, where $i = 1, 2$, is not zero. Hence (67) gives the invariants

$$\mathrm{Re}(z(-(-N + n, 0, j_i))z(-N, -m, j_i - s)z(n, m, s))), \tag{68}$$

where $z(-(-N + n, 0, j_i))$ and $z(-N, -m, j_i - s)$ for $i = 1, 2$ are known (see theorems 4 and 5). By lemma 1 the invariants (68) give the Fourier coefficient $z(n, m, s)$ under some zero measure conditions in the sense of remark 1. □

Thus, we considered the set of the polynomials of the form

$$p(x) = \sum_{a \in B(N,M,S) \cup C(\sqrt{N})} z(a) \mathrm{e}^{\mathrm{i}\langle a, x\rangle} \tag{69}$$

(see conditions 1 and 2, and theorems 4–6), where $B(N, M, S)$ and $C(\sqrt{N})$ are as defined in section 1 and in condition 2, respectively, and $z(a) \neq 0$. By $E(N, M, S)$ denote the subspace of $L_2(F)$ generated by functions $\mathrm{e}^{\mathrm{i}\langle a,x\rangle}$ for $a \in (B(N, M, S) \cup C(\sqrt{N}))$. Let $D(N, M, S)$ be the set of all polynomial of the form (69) satisfying the zero measure conditions, in the sense of remark 1, used in the proof of theorems 3–6. Due to remark 1, the set $D(N, M, S)$ is obtained from $E(N, M, S)$ by eliminating the sets whose n-dimensional measure is zero, where n is the number of the elements of $B(N, M, S) \cup C(\sqrt{N})$. Therefore, for every positive ε and for each $f_N \in E(N, M, S)$ the ball

$$\{h \in E(N, M, S) : \sup |h(x) - f_N(x)| < \varepsilon\}$$

contains an element p_N of $D(N, M, S)$, that is,

$$\sup_{x\in F} |p_N(x) - f_N(x)| < \varepsilon. \tag{70}$$

Now consider a triple sequence $\{(N_k, M_k, S_k)\}$ such that for all k the triple (N_k, M_k, S_k) satisfies the conditions which are satisfied for (N, M, S) (see condition 1) and $N_k \to \infty$ as $k \to \infty$. Thus, N_k, M_k, S_k are the prime numbers satisfying

$$M_k > 2N_k, \qquad S_k > 2M_k, \qquad N_1 \gg 1, \qquad \lim_{k\to\infty} N_k = \infty. \tag{71}$$

Denote by $D(N_k, M_k, S_k)$ the set obtained from $D(N, M, S)$ by substitution (N_k, M_k, S_k) for (N, M, S). Let

$$D = \cup_{k=1}^{\infty} D(N_k, M_k, S_k). \tag{72}$$

Theorem 7.

(a) The set D is dense in $W_2^s(F)$, where $s > 3$, in $\mathbb{C}^\infty$-topology.
(b) The invariants (2)–(4) determine constructively and uniquely, modulo inversion and translation (6), the potentials q of the set D.

Proof.

(a) Note that $f \in W_2^s(F)$ means that

$$f(x) = \sum_{a\in\Gamma} (f, \mathrm{e}^{\mathrm{i}\langle a,x\rangle})\mathrm{e}^{\mathrm{i}\langle a,x\rangle}, \qquad \sum_{a\in\Gamma} |(f, \mathrm{e}^{\mathrm{i}\langle a,x\rangle})|^2\, (1+|a|^{2s}) < \infty. \tag{73}$$

Without loss of generality, it can be assumed that $(f, 1) = 0$. If $s > 3$, then

$$\sup_{x\in F}\left|\sum_{a\in R(\sqrt{N})} (f, \mathrm{e}^{\mathrm{i}\langle a,x\rangle})\mathrm{e}^{\mathrm{i}\langle a,x\rangle}\right| \leqslant \sum_{a\in R(\sqrt{N})} |(f, \mathrm{e}^{\mathrm{i}\langle a,x\rangle})| = O((\sqrt{N})^{-(s-3)}), \tag{74}$$

where $R(\sqrt{N}) = \left\{a \in \Gamma : |a| \geqslant \frac{1}{2}\sqrt{N}\right\}$. It follows from the definitions of $B(N, M, S)$ and $C(\sqrt{N})$ that

$$\Gamma\backslash(B(N, M, S) \cup C(\sqrt{N}) \cup \{(0, 0, 0)\}) \subset R(\sqrt{N}). \tag{75}$$

By (74) and (75) $f(x)$ has an orthogonal decomposition $f(x) = f_N(x) + r_N(x)$, where

$$f_N(x) = \sum_{a\in(B(N,M,S)\cup C(\sqrt{N}))} (f, \mathrm{e}^{\mathrm{i}\langle a,x\rangle})\mathrm{e}^{\mathrm{i}\langle a,x\rangle}, \qquad \sup_{x\in F} |r_N(x)| = O((\sqrt{N})^{-(s-3)}), \tag{76}$$

$f_N \in E(N, M, S)$. Therefore, for any $\varepsilon > 0$ there exists N such that

$$\sup |f(x) - f_N(x)| < \varepsilon. \tag{77}$$

From (70) and (77) we obtain that for any $f \in W_2^s(F)$ and for any $\varepsilon > 0$ there exist N and $p_N(x) \in D(N, M, S)$ such that

$$\sup_{x \in F} |f(x) - p_N(x)| < 2\varepsilon,$$

which means that D is dense in $W_2^s(F)$ in the $\mathbb{C}^\infty$-topology.

(b) Let q be an element of D. Since the vector $(N_k, 1, 0)$ is a visible element of $\mathbb{Z}^3$ for each N_k, the invariants

$$\|q^{(N_k,1,0)}\|$$

for $k = 1, 2, \ldots$ (see (2)) are given. By the definition of D, the number

$$k =: \{\max s : \|q^{(N_s,1,0)}\| \neq 0\}$$

is finite. Therefore q belongs to the set $D(N_k, M_k, S_k)$. The statement of theorem $7(b)$ for this set follows from the definition of $D(N, M, S)$ and from theorems 4–6. $\square$

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